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Using the Laplace Transform to Solve the Volterra Integral Equation

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Abstract:

This paper presents study the Laplace transform of the convolution integral and use this concept to study the solution of the Volterra integral equation.
Keywords: Laplace Transform, Convolution, Integral Equation, Volterra Integral Equation.

حل معادلة فولتيرا باستخدام تحويل لابلاس

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الملخص:

تهدف هذه الورقة البحثية لدراسة تحويل لابلاس لتكامل الالتفاف و استخدام هذا المفهوم لحل معادلة فولتيرا التكاملية.

الكلمات المفتاحية: تحويل لابلاس ، الالتفاف ، المعادلة التكاملية ، معادلة فولتيرا التكاملية.

1. Introduction

Integral equations are very important in theoretical and applied researches.

Riemann's concept appeared and was strengthened in the tenth century and beyond by scholars such as Poincare and Hilbert. Integral equations research was reinforced by scholars such as Fredholm and Volterra, and in recent years it was deepened by equations. improper integration by scientists like Orison et al.

The various integral equations topics have grown and developed ; Because it is directly related to a large number of branches of mathematics, such as differential arithmetic, calculus of variations, approximation issues, optimal solutions, and boundary conditions, in addition to many issues with physical concepts and connections.

Here we present the Laplace transform as one of the most important integral transforms, which have many applications in scientific life and in engineering sciences. The Laplace transform also helps to solve related functions at intervals whose solutions can be obtained using traditional methods. It also changes the form of the original complex function to another form that is easier and simpler. In dealing with it by converting the differential equation into an algebraic equation that can be solved, and by finding the inverse Laplace transform, we get the solution to the original differential equation, so Laplace is used to solve some differential and complementary problems.

definition 1 . If $f(t)$ is a function defined for every real number $t \geq 0$, then the Laplace transform of this function is:

$$L\{f(t)\} = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad (1)$$

And using this definition, we conclude several laws:

$L(t^n) = \frac{(n)!}{s^{n+1}}$ $,s > 0$ $,L(\,t^p) = \frac{\Gamma(p+1)}{s^{p+1}},\ s > 0\,, p > -1$ $,L(\,e^{at}\,) = \frac{1}{s-a}$ $,s > a\,, L(\sin at) = \frac{a}{s^2+a^2}$ $,s > 0$ $, L(\cos at) = \frac{s}{s^2+a^2}$ $,s > 0.$

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Definition 2 . The gamma function Γ for any number x > 0 is defined as follows

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Definition 3 . step function knowing as follows

$$H_a(t) = H(t-a) = \begin{cases} 1 & ,t \geq a \\ 0 & ,t < a \end{cases}$$

It is called the Heaviside unit step function or simply the Heaviside function or the unit step function. This function is often useful in mathematical problems and applications.

Theorem 1. The Laplace transform is a linear operator

$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$

Theorem 2. (first shifting). If $f(t)$ is a continuous function and $L\{f(t)\} = F(s)$, then

$$L(e^{at}f(t)) = F(s-a)$$

Theorem 3. (second shifting). If $F(s) = L\{f(t)\}$ exists for values $s > b > 0$ and if a is a positive constant, then

$$L\{H(t-a)f(t-a)\} = e^{-as}L\{f(t)\} = e^{-as}F(s) \quad , \quad s > b$$

Theorem 4. (the iterative derivative). If $f(t)$ is a continuous function and $L\{f(t)\} = F(s)$, then

$$Lt^n f(t) = (-1)^n \frac{d^n}{ds^n}F(s)$$

Theorem 5. (time derivative). If $f(t)$ is a continuous function and $L\{f(t)\} = F(s)$, then

$$Lf'(t) = sF(s) - f(0)$$

$$Lf''(t) = s^2F(s) - sf(0) - f'(0)$$

$$Lf^{(n)}(t) = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$$

Theorem 6. If $f(t)$ is a continuous function and $L\{f(t)\} = F(s)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$$

$$L^{-1}\left\{\int_s^\infty F(s)ds\right\} = \frac{f(t)}{t}$$

Theorem 7. If f(t) is a continuous function and L{f(t)}=F(s), then

$$L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

$$L^{-1}\frac{F(s)}{s} = \int_0^t f(t)dt$$

definition 4. The integral equation is the equation in which the function to be determined is under the sign of integration, and the integral equation is called linear if the variable to be set is of first degree and is written in the form

$$\mu(x)y(x) = f(x) + \lambda \int_a^{b,x} K(x,t)y(t)dt \qquad (2)$$

Where $K(x,t)$ is the kernel of the integral equation (kernal), the upper limit of integration may be variable or constant, and the function $y(t)$ is the function to be determined, and the linear integral equation is called homogeneous if $f(x) = 0$, If the upper limit of integration is a fixed number and it is b , then the integral equation is called **Fredholm Integral Equation**, and if the variable upper term is x, then the equation is called **Volteraa**

Integral Equation. $\mu(x) = 0$, (2) It is called an integral equation of the first kind, but if $\mu(x) = 1$ it is called (2) an integral equation of the second kind, and if $\mu(x)$ is a function of x then it is called (2) an integral equation of the third kind.

2. convolution

The importance of convolution is evident in solving the integral and differential equations with initial conditions, and this concept has a special importance in many engineering fields and programs on the computer.

Definition 5. the convolution is defined as if we have the functions $g(t)$, $f(t)$ defined in the interval $[0, t]$ then the convolution of the functions g , f is defined as the function $h(t)$ as follows:

$$h(t) = \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du \qquad (3)$$

written by

$$h(t) = (f * g)(t).$$

It is called the convolution of functions f, g

Theorem 8.

If $L^{-1}\{F(s)\} = f(t)$ *and* $L^{-1}\{G(s)\} = g(t)$ *then*

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)du$$

$$= f * g$$

When $f * g$ is called the convolution of g, f .

Equivalent to that we have

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$$L\{f * g\} = L\{\int_0^t f(u)g(t-u)du\} = F(s)G(s)$$

This theory is called the convolution theorem.

proof

We have $L^{-1}\{G(s)\} = g(t)$

So the

$$G(s) = L\{g(t)\} = \int_0^\infty e^{-st} g(t)dt$$

Multiplying by F(s) we get

$$F(s)G(s) = \int_0^\infty e^{-st} g(t)F(s)dt$$

And since it is known that the variable of integration is optional, then t can be replaced by the variable u , so the

$$F(s)G(s) = \int_0^\infty [e^{-us}F(s)]g(u)du \qquad (4)$$

$$\because e^{-us}f(s) = L\{H(t-u)f(t-u)\}$$

$$= \int_0^\infty e^{-st} H(t-u)f(t-u)dt \qquad (5)$$

So, by substituting $e^{-us}F(s)$ from equation (5) into equation (4) we get

$$F(s)G(s) = \int_0^\infty g(u)[\int_0^\infty e^{-st} H(t-u)f(t-u)\, dt]du$$

$$= \int_0^\infty g(u)[\int_0^u e^{-st} H(t-u)f(t-u)dt + \int_u^\infty e^{-st} H(t-u)f(t-u)dt]du$$

$$\because H(t-u) = \begin{cases} 0 & , \ t < u \\ 1 & , \ t > u \end{cases}$$

So we note here that the limits of integration are $u \le t < \infty$, where $u > 0$.

If we change the order of the integration variables to become $dudt$ instead of $dtdu$, we find that u changes from $0 \ to \ t$ while t changes from $0 \ to \ \infty$, and so we get

$$F(s)G(s) = \int_0^\infty e^{-st}[\int_0^t f(t-u)g(u)du]dt$$

or

$$F(s)G(s) = L\{\int_0^t f(t-u)g(u)du\}$$

By taking the inverse Laplace of both sides

$$\therefore L^{-1}\{F(s)G(s)\} = \int_0^t f(t-u)g(u)du$$

Example 1.

find $L\{\int_0^t e^u \, sin(t-u) \, du\}$

solution

$g(t) = sint \, , f(t) = e^t$, by use theorem (8)

$$\therefore L\{\int_0^t e^u \, sin(t-u) \, du = L\{e^t\} \, . \, L\{sin \, t\} = \frac{1}{s-1} \cdot \frac{1}{s^2+1}$$

$$= \frac{1}{(s-1)(s^2+1)}$$

Theorem 9.

Convolution is a commutative operation

$$f(t) * g(t) = g(t) * f(t)$$

proof

establish compensation

$$y = t - u \ \rightarrow \frac{dy}{du} = -1 \ \rightarrow dy = -du$$

$$u = 0 \ \rightarrow y = t\ , u = t \ \rightarrow y = 0$$

$$f(t) * g(t) = \int_0^t [f(t-u)g(u)]du$$

$$f(t) * g(t) = -\int_t^0 [g(t-y)f(y)]dy$$

$$= \int_0^t [g(t-y)f(y)]dy = g(t) * f(t)$$

Example 2.

Find the inverse Laplace transform of the two functions

$$(i) \quad \frac{1}{s^2(s^2+1)} \quad , \quad (ii) \quad \frac{1}{s^2-1}$$

solution

$$(i) \qquad \because L\{t\} = \frac{1}{s^2} \quad , \quad L\{\sin t\} = \frac{1}{s^2+1}$$

So based on theorem (8), then: $L^{-1}\left\{\frac{1}{S^2(S^2+1)}\right\}$ is the bypass of the two functions $\sin t$, t

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (t-u)\sin u\, du = t\int_0^t \sin u\, du - \int_0^t u \sin u\, du$$

$$= t - t\cos t + t\cos t - \sin t$$

$$= t - \sin t$$

This result can also be obtained in the following way:

Using partial fractions

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Using partial fractions, and since L^{-1} is a linear operator, then:

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= t - \sin t$$

(ii) $$\frac{1}{s^2-1} = \frac{1}{s-1}\cdot\frac{1}{s+1}$$

$$\because L^{-1}\left\{\frac{1}{s-1}\right\} = e^t \quad و \quad L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$\therefore L^{-1}\left\{\frac{1}{s^2-1}\right\} = \int_0^t e^{t-u}e^{-u}du = e^t\int_0^t e^{-2u}du$$

$$= e^t \left[\frac{e^{-2u}}{-2}\right]_0^t$$

$$= \frac{1}{2} e^t (1 - e^{-2t})$$

$$L^{-1}\left\{\frac{1}{s^2 - 1}\right\} = \frac{1}{2}(e^t - e^{-t})$$

This result can also be obtained using partial fractions

$$\frac{1}{s^2 - 1} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$$

L^{-1} is a linear operator, then

$$L^{-1}\left\{\frac{1}{s^2 - 1}\right\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}\right]$$

$$= \frac{1}{2}(e^t - e^{-t})$$

3. Solve the Volterra equation using the Laplace transform

The Laplace transformer is considered one of the most powerful transformers that are useful in solving the integral equations arising from the differential equations with initial conditions, which is what we call the Volterra integral equation, which takes the position

$$\mu y(x) = f(x) + \lambda \int_0^x k(x - t) y(t) dt \qquad (6)$$

When the kernel of the integral equation (6) is difference kernel, to solve equation (6) using the Laplace transform, we follow the following:

$$\mu\, L\{y(x)\} = L\{\mathrm{f}(x)\} + \lambda L\left\{\int_0^x k(x-t)y(t)d(t)\right\}$$

From the concept of the convolution theorem, it can be written in the form:

$$\mu L\{y(x)\} = L\{\mathrm{f}(x)\} + \lambda L\{(k * y)\}$$

$$\rightarrow \mu Y(s) = \mathrm{F}(s) + \lambda\, K(s)\, Y(s)$$

$$\therefore\ Y(s) = \frac{\mathrm{F}(s)}{\mu - \lambda\, K(s)}.$$

Taking the Laplace inverse of both sides, we get the solution to the equation

$$y(x) = L^{-1}\{\frac{F(s)}{\mu - \lambda\, K(s)}\}$$

Example 3.

Find a solution to the following Volterra integral equation:

$$y(x) = x^2 + \int_0^x \sin(x-t)y(t)dt$$

solution

By taking Laplace to both sides

$$y(s) = \frac{2}{s^3} + \frac{1}{s^2+1}\, y(s)$$

We put the y(s) on one side and the rest on the other side

$$\left(1-\frac{1}{s^2+1}\right)y(s)=\frac{2}{s^3}$$

$$\frac{s^2}{s^2+1}y(s)=\frac{2}{s^3}$$

To get y(s), we divide by a coefficient

$$y(s)=\frac{2(s^2+1)}{s^5}$$

$$y(s)=\frac{2s^2}{s^5}+\frac{2}{s^5}$$

$$y(s)=\frac{2}{s^3}+\frac{2}{s^5}$$

By taking L^{-1} on both sides

$$\to y(x)=L^{-1}\left\{\frac{2}{s^3}\right\}+L^{-1}\left\{\frac{2}{s^5}\right\}$$

$$y(x)=x^2+\frac{1}{12}x^4$$

Example 4.

Find the solution to the following Volterra integral equation

$$y(x)=\sin x+2\int_0^x \cos(x-t)y(t)dt$$

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Solution

$$y(s) = \frac{1}{s^2+1} + 2\left(\frac{s}{s^2+1}\right)y(s)$$

$$\left(1 - \frac{2s}{s^2+1}\right)u(s) = \frac{1}{s^2+1}$$

$$\left(\frac{s^2-2s+1}{s^2+1}\right)y(s) = \frac{1}{s^2+1}$$

$$y(s) = \frac{1}{s^2-2s+1}$$

$$y(s) = \frac{1}{(s-1)^2}$$

$$\rightarrow y(x) = L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$\therefore y(x) = xe^x$$

Example 5.

Find the solution to the following Volterra integral equation

$$y(x) = e^x - \cos x - 2\int_0^x e^{x-t}y(t)dt$$

solution

$$y(s) = \left(\frac{1}{s-1} - \frac{s}{s^2+1}\right) - 2\left(\frac{1}{s-1}\right)y(s)$$

$$\{1 - (-2)\left(\frac{1}{s-1}\right)\}y(s) = \frac{1}{s-1} - \frac{s}{s^2+1}$$

$$\left(\frac{s+1}{s-1}\right)y(s) = \frac{1}{s-1} - \frac{s}{s^2+1}$$

$$y(s) = \frac{1}{s^2+1}$$

$$\rightarrow y(x) = L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\therefore y(x) = \sin x$$

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